# GROUPS OF HOMOMORPHISM GRADED BY $G$-SETS

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## 1. Introduction

Let $G$ be a group and $R = \bigoplus_{g\in G} R_g$ a $G$-graded ring. If $M = \bigoplus_{g\in G} M_g$ and $N = \bigoplus_{g\in G} N_g$ are $G$-graded rings then E.C. Dade showed in [D1] that the group $\mathrm{Hom}_R(M,N)$ has a subgroup denoted $\mathrm{HOM}_R(M,N)$ which can be endowed with a natural $G$-grading:

$$\begin{aligned}\mathrm{HOM}_R(M,N)_g &= \{f \in \mathrm{Hom}_R(M,N) \mid f(M_h) \subseteq N_{hg} \text{ for all } h \in G\}\\ &= \mathrm{Hom}_{R-gr}(M, N(g)),\end{aligned}$$

where $R$-gr denotes the category of $G$-graded $R$-modules and grade preserving $R$-morphisms and $N(g)$ is the $G$ graded module, called the $g$-suspension of $N$, with $N(g) = N$ and $N(g)_h = N_{hg}$ for all $h \in G$.

In particular, $\mathrm{END}_R(M) = \mathrm{HOM}_R(M,M)$ is a $G$-graded ring and $\mathrm{HOM}_R(M,N)$ becomes a $G$-graded $E$-module, where $E = \mathrm{END}_R(M)^{opp}$ (with grading given by $E_g = \mathrm{END}_R(M)_{g^{-1}}$).

The natural problem which arises is to give condition under which $\mathrm{HOM}_R(M,N) = \mathrm{Hom}_R(M,N)$. Some condition were established by Dade in [D1] and a general answer was given by Gómez-Pardo, Militaru and Năstăsescu in [GMN], using among other things the fact, proved in [GMN, Theorem 1.2], that $\mathrm{Hom}_R(M,N)$ is the completion of $\mathrm{HOM}_R(M,N)$ in the finite topology.

Dade also introduced in [D2] a useful generalization of $G$-graded modules, namely modules graded by $G$-sets. We shall be interested only in modules graded by transitive $G$-sets, so let $H$ be a subgroup of $G$ and denote by $G/H$ the set $\{gH \mid g \in G\}$ of left cosets of $H$ in $G$. An $R$-module $N$ is $G/H$ if $N = \bigoplus_{x\in G/H} N_x$ (as additive

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subgroups) and $R_gN_x \subseteq N_{gx}$ for all $g \in G$ and $x \in G/H$. Denote by $(G/H,R)$-gr the category of $G/H$-graded $R$-modules and grade preserving $R$-morphisms. Observe that $(G/G,R)$-gr$=R$-mod, while $(G/1,R)$-gr $= R$-gr. Moreover, $(G/H,R)$-gr is a Grothendieck category and it was investigated in detail in [NRV] and [NSV].

The construction of "HOM" was generalized in [M, section2]. If $M$ is a $G$-graded $R$-module and $N$ is a $G/H$-graded $R$-module then $\mathrm{Hom}_{G/H,R}(M,N)$ has a subgroup $\mathrm{HOM}_R(M,N)$ endowed with a natural $G/H$-grading such that $\mathrm{Hom}_{G/H,R}(M,N)$ is a $G/H$-graded $R$-module.

Recall that if $K \le H \le G$ then there is a functor

$$\mathcal{U} = \mathcal{U}^{G/H}_{G/K} \colon (G/K,R)\text{-gr} \to (G/H,R)\text{-gr}$$

sending a $G/K$-graded $R$-module $N$ to the $G/H$-graded $R$-module $\mathcal{U}(N) = N$ with

$$\mathcal{U}(N)_x = \bigoplus_{\substack{y\in G/K\\ y\subseteq x}} N_y.$$

The details on this functor and on its right adjoint are given in Section 2.

In Section 3 we recall the definition of $\mathrm{HOM}_{G/K,R}(M,N)$ and we investigate its relation to $\mathrm{HOM}_{G/H,R}(M,N)$ in connection with the finite topology on $\mathrm{Hom}_R(M,N)$. Our situation gives rise to two composite functor:

$$\mathrm{HOM}_{G/H,R}(M,-) \circ \mathcal{U}^{G/K}_{G/H} \colon (G/K,R)\text{-gr} \to (G/H,R)\text{-gr},$$

and

$$\mathcal{U}^{G/K}_{G/H} \circ \mathrm{HOM}_{G/K,R}(M,-) \colon (G/K,R)\text{-gr} \to (G/H,R)\text{-gr}.$$

Loosely speaking we ask when these functors are equal. Our main result in Section 4 generalize [GMN, Th. 3.4] and states that if $H/K$ is infinite, then the functors are equal if and only if $M$ is a small $R$-module.

In this paper rings are always associative with unit element, and modules are unitary and left. Besides the above mentioned papers, we refer to [NRV] for general facts on graded rings and modules.

## 2. The grade forgetting functor and its adjoint

In this section we provide the details on the properties of the grade forgetting functor.

Let $R = \bigoplus_{g\in G} R_g$ be a $G$-graded ring as in the introduction, and fix two subgroups $K \le H$ of $G$.

**2.1.** The grade forgetting functor

$$\mathcal{U} = \mathcal{U}^{G/K}_{G/H} \colon (G/K,R)\text{-gr} \to (G/H,R)\text{-gr}$$

is defined as follows: for $M=\bigoplus_{x\in G/K} M_x \in (G/K,R)\text{-gr}$ we have $\mathcal{U}(M)=\bar{M}=\bigoplus_{y\in G/H}\bar{M}_y$, where $\bar{M}=M$ (as $R$-module), and $\bar{M}_y=\bigoplus_{x\subseteq y} M_x$ for all $y\in G/H$, and obviously, $\mathcal{U}(f)=f$ for every morphism $F\colon M\to M'$ in $(G/K,R)\text{-gr}$.

**2.2.** There is functor in the opposite direction

$$\mathcal{F}=\mathcal{F}_{G/H}^{G/K}\colon (G/H,R)\text{-gr}\to (G/K,R)\text{-gr}$$

defined as follows: for $N=\bigoplus_{y\in G/H} N_y\in (G/H,R)\text{-gr}$ we have

$$\mathcal{F}(N)=\tilde{N}=\bigoplus_{x\in G/K}\tilde{N}_x,\qquad \tilde{N}_x=N_{xH},$$

with multiplication by scalars given by $r_g\tilde{n}_x=r_gn_y\in N_gx$ where $y=xH$, $\tilde{n}_x=n_y\in N_y$, $r_g\in R_g$, $g\in G$.

If $f\colon N\to N'$ is morphism in $(G/H,R)\text{-gr}$, then $\tilde{f}=\mathcal{F}(f)\colon\tilde{N}\to\tilde{N}'$ is given by $\tilde{f}(\tilde{n}_x)=f(n_y)\in\tilde{N}_x=N_y$, with $y=xH$ and $\tilde{n}_x=n_y$ as above.

**2.3. Lemma.** *With the above notation we have:*

*a) $\mathcal{F}=\mathcal{F}_{G/H}^{G/K}$ is a right adjoint of $\mathcal{U}=\mathcal{U}_{G/H}^{G/K}$*

*b) If $H/K$ is finite then $\mathcal{F}$ is also a left adjoint of $\mathcal{U}$.*

*Proof.* Let $M\in(G/K,R)\text{-gr}$ and $N\in(G/H,R)\text{-gr}$. We define the functorial isomorphism

$$\Phi_{M,N}\colon \operatorname{Hom}_{(G/H,R)\text{-gr}}(\mathcal{U}(M),N)\to\operatorname{Hom}_{(G/K,R)}(M,\mathcal{F}(N)),$$

by letting, for $f\colon\mathcal{U}(M)\to N$ and $m_x\in M_x$,

$$\Phi_{M,N}(f)(m_x)=f(m_x)\in\mathcal{F}(N)_x=N_{xH}.$$

Clearly, $\Phi_{M,N}$ is a well defined group homomorphism.

Define further

$$\Psi_{M,N}\colon \operatorname{Hom}_{(G/K,R)\text{-gr}}(M,\mathcal{F}(N))\to\operatorname{Hom}_{(G/H,R)}(\mathcal{U}(M),N),$$

by letting, for $g\colon M\to\mathcal{F}(N)$ and $m_x\in M_x=\mathcal{U}(M)_{xH}$,

$$\Psi_{M,N}(g)(m_x)=g(m_x)\in\mathcal{F}(N)_x=N_{xH}.$$

Again $\Psi_{M,N}$ is well defined and it is obvious that $\Psi_{M,N}=\Phi_{M,N}^{-1}$.

b) Assume that $H/K$ is finite, and for $M$ and $N$ as above define

$$\Theta_{N,M}\colon \operatorname{Hom}_{(G/K,R)\text{-gr}}(\mathcal{F}(N),M)\to\operatorname{Hom}_{(G/H,R)}(N,\mathcal{U}(M)),$$

and for $f\colon \mathcal{F}(M)\to N$ and $n_y\in N_y$,

$$\Theta_{N,M}(f)(n_y)=\sum_{\substack{x\in G/K\\ x\subseteq y}} f(\tilde{n}_x),$$

with $\tilde{n}_x=n_y\in\mathcal{F}(N)_x$.

Conversely, define

$$\Gamma_{N,M}\colon \mathrm{Hom}_{(G/H,R)\text{-gr}}(N,\mathcal{U}(M))\to \mathrm{Hom}_{(G/K,R)}(\mathcal{F}(N),M),$$

by letting, for $g\colon N\to\mathcal{U}(M)$ and $\tilde{n}_x=n_y\in\mathcal{F}(N)_x=N_{xH}$, $y=xH$

$$\Gamma_{N,M}(g)(\tilde{n}_x)=g(n_x)_x,$$

where $g(\tilde{n}_x)$ is the $x$-th component of $g(\tilde{n}_x)\in\bigoplus_{x\in G/K}M_x$. Again it is easy to check that $\Gamma_{N,M}=\Theta_{N,M}^{-1}$.

*2.4. Remark.* a) The unit $\zeta$ of the adjoint pair $(\mathcal{U},\mathcal{F})$ is defined by

$$\zeta_M\colon M\to\mathcal{F}(\mathcal{U}(M)),\quad \zeta_M(m_x)=m_x\in\mathcal{F}(\mathcal{U}(M))_x,$$

for all $x\in G/K$, $m_x\in M_x$, the counit $\xi_N$ given by

$$\xi_N\colon\mathcal{U}(\mathcal{F}(N))\to N,\quad \xi_N(\tilde{n}_x)=\tilde{n}_x\in N_{xH},$$

for all $\tilde{n}_x=n_y\in N_y$, $x\in G/K$, $y=xH$

b) If $H/K$ is finite then the unit $\eta_N$ of the adjoint pair $(\mathcal{F},\mathcal{U})$ is defined by

$$\eta_N\colon N\to\mathcal{U}(\mathcal{F}(N)),\quad \eta_N(n_y)=\sum_{\substack{x\in G/K\\ x\subseteq y}}\tilde{n}_x,$$

where $\tilde{n}_x=n_y\in N_y$. The counit $\mu_M$ is defined by

$$\mu_M\colon\mathcal{F}(\mathcal{U}(M))\to M,\quad \mu_M({}^{x'}m_x)=\begin{cases} m_x, & \text{if } x=x'\\ 0, & \text{if } x\neq x',\end{cases}$$

where $x,x'\in G/K$, $xH=x'H$ and ${}^{x'}m_x\in\mathcal{F}(\mathcal{U}(M))_{x'}$.

Observe that $\mu_M\circ\zeta_M=id_M$ and $\xi_N\circ\eta_N=\mid H/K\mid id_N$.

c) Recall that if $(N_i)_{i\in I}$ is a family of objects of $(G/H,R)$-gr then the direct sum of this family is $\bigoplus_{i\in I}N_i$ with $G/H$-grading given by $(\bigoplus_{i\in I}N_i)_y=\bigoplus_{i\in I}(N_i)_y$. It follows that $(G/K,R)$-gr is an AB3 category, and by its construction, it easily follows that $\mathcal{F}_{G/H}^{G/K}$ preserves direct sum.

**2.5.** An another functor which we will use is the "suspension" functor:

$$\mathcal{S}_G/H^g\colon (G/H,R)\text{-gr}\to(G/{}^gH)\text{-grdefined by } \mathcal{S}_G/H^g(N)=N(g), g\in G,$$

where ${}^gH=gHg^{-1}$ and $N(g)=\bigoplus_{x\in G/{}^gH}N(G)_x$ with $N(g)_x=N_{xg}$.

## 3. The finite topology on $\mathrm{HOM}_{G/K,R}(M,N)$

**3.1.** Let $R$ be a $G$-graded ring, $K$ a subgroup of $G$, $M = \bigoplus_{g\in G} M_g \in R\text{-gr}$, and $N = \bigoplus_{x\in G/K} N_x \in (G/K,R)\text{-gr}$. By [M, 2.9], for each $x \in G/K$, the set

$$\mathrm{HOM}_{G/K,R}(M,N)_x = \{f \in \mathrm{Hom}_R(M,N) \mid f(M_g) \subseteq N_{gx} \text{ for all} g \in G\}$$

is an additive subgroup of $\mathrm{Hom}_R(M,N)$. Moreover the sum

$$\mathrm{HOM}_{G/K,R}(M,N) = \sum_{x\in G/K} \mathrm{HOM}_{G/K,R}(M,N)_x$$

is direct. Remark that we also have the equality

$$\mathrm{HOM}_{G/K,R}(M,N) = \mathrm{Hom}_{(G/{}^xK,R)\text{-gr}}(\mathcal{U}^{G/1}_{G/{}^xK}(M), N(x)),$$

where ${}^xK = xKx^{-1}$, $\mathcal{U}^{G/1}_{G/{}^xK}\colon R\text{-gr} \to (G/{}^xK,R)\text{-gr}$ is the grade-forgetting functor and $N(x)$ is a object of $(G/{}^xK,R)\text{-gr}$ defined by

$$N(x) = N, \quad N(x)_y = N_{xy} \text{ for every } y \in G/{}^xK.$$

In particular, we have that $\mathrm{End}_R(M)$ contains a $G$-graded subring

$$\mathrm{END}_R(M) = \bigoplus_{g\in G} \mathrm{END}_R(M)_g,$$

with

$$\mathrm{END}_R(M)_g = \{f \in \mathrm{End}_R(M) \mid f(M_h) \subseteq M_{hg} \text{ for all } h \in G\},$$

and $\mathrm{HOM}_{G/K,R}(M,N)$ becomes a $G/K$-graded $E$-module where $E = \mathrm{END}_R(M)^{opp}$.

The aim of this section is to establish the relationship between $\mathrm{HOM}_{G/K,R}(M,N)$ and $\mathrm{Hom}_R(M,N)$ in topological terms.

**3.2.** Let $\mathcal{A}$ be a Grothendieck category and let M,N be two object of $\mathcal{A}$. Recall that the finite topology on $\mathrm{Hom}_R(M,N)$ is defined by giving a basis for the filter of the neighborhoods of $0$ as follows:

$$\mathcal{F}(0) = \{V(0,X) \mid X \text{ ranges over the finitely generated subobjects of } M\},$$

where

$$V(0,X) = V_{M,N}(0,X) = \{f \in \mathrm{Hom}_{\mathcal{A}}(M,N) | X \subseteq \mathrm{Ker}\, f\}.$$

In this way, $\mathrm{Hom}_{\mathcal{A}}(M,N)$ becomes a topological abelian group, which is actually a subspace of $N^M$ endowed with the product topology, each of the components having the discrete topology.

If $\mathcal{A}$ is locally finitely generated (that is, it has a set of finitely generated generators or, equivalently, every object is the sum of its finitely generated subobject), then $\mathrm{Hom}_{\mathcal{A}}(M,N)$ is a complete Hausdorff topological space in the finite topology (see [GMN, Section 2]). In particular this hold if $\mathcal{A}=R$-mod. In this case, a filter of neighborhoods of 0 is

$$\{V(0,X) \mid X \text{ is a finite subset of } M\}.$$

Indeed, if $V(0,\{x_1,\dots,x_n\})$ is a neighborhood of 0 in the finite topology, defined in this way, then $V(0,\{x_1,\dots,x_n\}) = V(0,\langle x_1,\dots,x_n\rangle)$ is a neighborhood of 0 in the finite topology; conversely, $V(0,X)=V(0,\{x_1,\dots,x_n\})$ where $\{x_1,\dots,x_n\}$ generates the submodule $X$ of $M$.

**3.3.** Returning to the case when $X\in R$-gr and $N\in (G/K,R)$-gr we have that $\mathrm{HOM}_{G/K,R}(M,N)\subseteq \mathrm{Hom}_R(M,N)$ is a topological space with the induced topology and this is precisely the finite topology. Indeed, if

$$V'(0,\{x_1,\dots,x_n\}) = \{f\in \mathrm{HOM}_{G/K,R}(M,N) \mid f(x_1)=\ldots=f(x_n)\}$$

is a neighborhood of 0 in the finite topology on $\mathrm{HOM}_{G/K,R}(M,N)$, then

$$V'(0,\{x_1,\dots,x_n\}) = V(0,\{x_1,\dots,x_n\})\cap \mathrm{HOM}_{G/K,R}(M,N),$$

where

$$V(0,\{x_1,\dots,x_n\}) = \{f\in \mathrm{Hom}_R(M,N) \mid f(x_1)=\ldots=f(x_n)\}.$$

Moreover, for every $f\in \mathrm{HOM}_{G/K,R}(M,N)$ we have that

$$V'(f,\{x_1,\dots,x_n\}) = f+V'(0,\{x_1,\dots,x_n\}).$$

Thus $\mathrm{HOM}_{G/K,R}(M,N)$ is a Hausdorff topological abelian group in the finite topology.

**3.4.** Recall that if $X$ is a topological space, then the family $\{x_i \mid i\in I\}$ is called summable to $x\in X$ if for every neighborhood $V$ of $x$ there is a finite subset $J_V$ of $I$ such that for each finite subset $J$ of $I$ containing $J_V$ we have $\sum_{i\in J} x_i\in V$. In this case we write $\sum_{I\in I} x_i = x$.

We claim that if $X=\mathrm{Hom}_R(M,N)$, then it is enough to consider only neighborhoods of the type $V(f,\{m\})$ where $f\in Hom_R(M,N)$ and $m$ is a homogeneous element of $M$. Indeed, let $\{f_i \mid i\in I\}$ be a family in $\mathrm{Hom}_R(M,N)$ with the property that for every neighborhood $V(f,m_g)$ with $m_g\in M_g$, there is a subset $J_0$ of $I$ such that for every finite subset $J$ of $I$ containing $J_0$ we have $\sum_{i\in J} f_i\in V(f,m_g)$. Now let $m=m_1+\ldots+m_n\in M$ where $m_1,\dots,m_n$ are homogeneous and let $J_1,\dots,J_n$ be finite subsets such that for every $j \in \{1, \ldots, \}$ and for every finite subset $J$ of $I$ with $J_j \in J$ we have $\sum_{i\in J} f_i \in V(f, m_j)$, that is $\sum_{i\in J} f_I(m_j) = f(m_j)$. If we let $J_0 = \bigcup_{j=1}^n J_j$, then $(\sum_{i\in J} f_i)(m_j) = f(m_j)$, hence $\sum_{i\in J} f_i \in V(f, m)$ for every finite subset $J$ of $I$ containing $J_0$. Finally let $V = V(f, \{m_1, \ldots, m_k\})$ be a neighborhood of $f$. Then there are the finite subsets $J'_1, \ldots, J'_k$, of $I$ such that for every $l \in \{1, \ldots, k\}$ and for every finite subset $J$ of $I$ containing $J'_l$ we have $(\sum_{i\in J} f_i)(m_l) = f(m_l)$. If we set $J_V = \bigcup_{l=1}^k J'_l$, then for every finite subset $J$ of $I$ containing $J_V$ we have $\sum_{i\in J} f_i \in V$, hence $\sum_{i\in I} f_i = f$, and the claim is proved.

The goal of this section is to show that $\mathrm{Hom}_R(M, N)$ is the completion of its subgroup $\mathrm{HOM}_{G/H,R}(M, N)$, in the finite topology and to and to examine the relationship between $\mathrm{HOM}_{G/H,R}(M, N)$ and $\mathrm{HOM}_{G/K,R}(M, N)$, where $K \leq H$ are subgroups of $G$.

**3.5.** Let $f \in \mathrm{Hom}_R(M, N)$, $g \in G$ and $x \in G/H$. We define the abelian groups homomorphism

$$f_x \colon M = \bigoplus_{g\in G} M_g \to N$$

with components

$$f_x^g \colon M_g \to N, \quad f_x^g(m_g) = f(m_g)_{gx} \text{ for all } m_g \in M_g,$$

where $f(m_g)_{gx}$ is the homogeneous component of $f(m_g) \in N = \bigoplus_{y\in G/H} N_y$ belonging to $N_{gx}$.

If $r_h \in R_h$, $h \in G$, then $r_h m_g \in M_{hg}$, so

$$f_x(r_h m_g) = f_x^{hg}(r_h m_g) = f(r_h m_g)_{hgx} = (r_h f(m_g))_{hgx}.$$

Since every element of $N$ has a unique decomposition into homogeneous components, it follows that $f_x(r_h m_g) = r_h f_x(m_g)$. Consequently, $f_x$ is $R$-linear. Further, by definition we have that $f_x \in \mathrm{HOM}_{G/H,R}(M, N)_x$.

Now we may state the main result of this section.

**3.6. Theorem.** *If $M \in R$-gr and $N \in (G/H, R)$-gr, then with the above notations the following statements hold:*

*a) The family $\{f_x \mid x \in G/H\}$ is summable to $f$ in the finite topology, and the components $f_x$ are uniquely determined by the properties $f_x \in \mathrm{HOM}_{G/H,R}(M, N)_x$ and $\sum_{x\in G/H} f_x = f$.*

*b) $\mathrm{Hom}_R(M, N)$ is the completion of $\mathrm{HOM}_{G/H,R}(M, N)$ in the finite topology.*

*Proof.* a) By (3.4) we may restrict to the case of neighborhoods of the form $V(f, m_g)$ where $m_g \in M_g$ for some $g \in G$. Let $f(m_g) = n_{x_1} + \ldots + n_{x_k}$ with $n_{x_j} \in N_{x_j}$ be the decomposition of $f(m_g)$ into homogeneous components. We set $J_0 = \{x_1, \dots x_k\}$. Then $J_0$ is a finite subset of $G/H$ and $f(m_g) = (\sum_{x\in J_0} f_x(m_g)$, hence $\sum_{x\in J_0} f_x \in V(f, m_g)$. Since for each $x \in G/K \setminus J_0$ we have $f(m_g) = 0$, it follows that for every finite subset $J$ of $I$ containing $J_0$, $f(m_g) = (\sum_{x\in J} f_x(m_g)$. Therefore $\sum_{x\in J} f_x \in V(f, m_g)$, and consequently, $\sum_{x\in G/H} f_x = f$ in the finite topology.

For the uniqueness, we assume that $g_x \mid x \in G/H$ is another family of morphisms such that $g_x \in \mathrm{HOM}_{G/H,R}(M,N)_x$ and $\sum_{x\in G/H} g_x = f$. If $f_{x_0} \neq g_{x_0}$ for some $x_0 \in G/H$, then there exits $m_g \in M_g$ such that $f_{x_0} \neq g_{x_0}$. We consider the neighborhood $V(f, m_g)$ of $f$ in the finite topology. Then we may find the finite subset $J_0 \in \mathcal{P}_f(G/H)$ (where $\mathcal{P}_f(G/H)$ denotes the set of all finite subsets of $G/H$) with the property that for every $J \in \mathcal{P}_f(G/H)$ for which $J_0 \subseteq J$ we have $\sum_{x\in G/H} f_x \in V(f, m_g)$ and $\sum_{x\in G/H} g_x \in V(f, m_g)$. If we set $J = J_0 \cup \{x_0\}$, then $J \in \mathcal{P}_f(G/H)$ and $(\sum_{x\in J} f_x(m_g) = f(m_g) = (\sum_{x\in J} g_x(m_g)$. By the uniqueness of the decomposition into homogeneous elements it follows that $f_{x_0}(m_g) = g_{x_0}(m_g)$.

b) Let $f$ be an element of $\mathrm{Hom}_R(M,N)$. With the above notations, for some $J \in \mathcal{P}_f(G/H)$ we have

$$\sum_{x\in J} f_x \in \bigoplus_{x\in G/H} \mathrm{HOM}_{G/H,R}(M,N)_x = \mathrm{HOM}_{G/H,R}(M,N),$$

and a) implies that $\mathrm{HOM}_{G/H,R}(M,N) \cap V(f,m) \neq 0$. Hence $\mathrm{HOM}_{G/H,R}(M,N)$ is dense in $\mathrm{Hom}_R(M,N)$ in the finite topology. But $\mathrm{Hom}_R(M,N)$ is a complete Hausdorff topological space in the finite topology, hence it is the completion of $\mathrm{HOM}_{G/H,R}(M,N)$.

**3.7. Corollary.** *If $K \le H$ are subgroups of $G$, then $\mathrm{HOM}_{G/K,R}(M,N)$ is dense in $\mathrm{HOM}_{G/H,R}(M,N)$ in the finite topology.*

**3.9. Proposition.** *With the notations from the prop. 3.8., if H/K is a finite set then*

$$\mathrm{HOM}_{G/H,R}(M,N) = \mathrm{HOM}_{G/K,R}(M,N)$$

.

*Proof.* For begin let $f \in \mathrm{HOM}_{G/H,R}(M,N)_{gH}$ and $\{h_1, \dots, h_n\} = [H/K]$ be a transversely for $H/K$. Since $H = \bigcup_{h\in[H/K]} hK$ we have, for a fixed $gH \in G/H$, $gH = \bigcup_{h\in[H/K]} ghK = gh_1K \cup \ldots \cup gh_nK$, and $G/K = \{ghK \mid g \in [G/H], h \in [H/K]\}$. Thus

$$N = \bigoplus_{x\in G/K} N_x = \bigoplus_{g\in[G/K]} (\bigoplus_{h\in[H/K]} N_{ghK}) = \bigoplus_{g\in[G/H]} (\bigoplus_{i=1}^{n} N_{gh_iK}) = \bigoplus_{i=1}^{n} (\bigoplus_{g\in[G/H]} N_{gh_iK}).$$

We consider the canonical projections $p_n\colon N \to \bigoplus_{g\in[G/H]} N_{gh_iK}$ and the composition $p_n\circ f\colon M \to \bigoplus_{g\in[G/H]} N_{gh_iK}$. Since $f(M_k) \subseteq N_{kgH}$ and for each $kH \in G/H$ there exist a unique $i \in \{1,\dots,n\}$ such that $gh_iK \subseteq kgH$. Thus $f$ may be write as $\sum_{i=1}^{n}(f\circ p_n)$ with $f\circ p_n \in \mathrm{HOM}_{G/K,R}(M,N)$. Now the general case when $f \in \mathrm{HOM}_{G/H,R}(M,N)$ follow from the fact that $f$ may be write as a finite sum of morfisms belonging to $\mathrm{HOM}_{G/H,R}(M,N)_{gH}$ with $gH$ ranges over $G/H$.

## 4. Small graded modules

**4.1.** Let $\mathcal{A}$ be an abelian category satisfying AB3, and let $M$ be an object of $\mathcal{A}$. $M$ is called *small* if the functor $\mathrm{Hom}_{\mathcal{A}}(M,-)\colon \mathcal{A}\to Ab$ preserves direct sum or, equivalently, preserves denumerable direct sum [S, p. 134]. This is equivalent to the fact that any morphism $f\colon M \to \bigoplus_{i\in\mathbb{N}} X_i$ factors trough a finite coproduct $\bigoplus_{i\in F} X_i$, where $F$ a finite subset of $\mathbb{N}$ [Mitchell, pag.74]. More generally we say that $M$ is $N$-*small* if $\mathrm{Hom}_{\mathcal{A}}(M,-)\colon \mathcal{A}\to Ab$ preserves (denumerable) direct sum of copies of $N$ (for $N=M$ one gets the concept of a *self-small* object). We will use the following results.

**4.2. Proposition.** [GMN, Proposition 1.1] *Let $\mathcal{A}$ be an AB3 category and $M$ an object of $\mathcal{A}$. Then $M$ is small if and only if $M$ is $N$-small for every object $N$ of $\mathcal{A}$.*

**4.4. Theorem.** [GMN, th. 1.3.] *Let $\mathcal{A}$ and $\mathcal{B}$ be AB3 categories, $M\in\mathcal{A}$ and $N\in\mathcal{B}$. Let $\mathcal{U}\colon\mathcal{A}\to\mathcal{B}$ be a left adjoint of the functor $\mathcal{F}\colon\mathcal{B}\to\mathcal{A}$ and assume that $\mathcal{F}$ preserves direct sum. Then the following assertion hold:*

*a) $M$ is $\mathcal{F}(N)$-small in $\mathcal{A}$ if and only if $\mathcal{U}(M)$ is $N$-small in $\mathcal{B}$.*

*b) If $M$ is small in $\mathcal{A}$ then $\mathcal{U}(M)$ is small in $\mathcal{B}$.*

**4.5.** The connection between small objects and finite topology is as follows (cf. [GMN, pp. 3176-3177]). If the finite topology of $\mathrm{Hom}_{\mathcal{A}}(M,N)$ is discrete, where $\mathcal{A}$ is a Grothendieck category, then $M$ is $N$-small. The converse is true if we assume in addition that $\mathcal{A}$ being locally finitely generated and the finite topology of $\mathrm{Hom}_{\mathcal{A}}(M,N)$ being first countable. If $\mathcal{A}$ is a locally finitely generated Grothendieck category and $M\in\mathcal{A}$ is countably generated, then $M$ is $N$-small if and only if the finite topology of $\mathrm{Hom}_{\mathcal{A}}(M,N)$ is discrete.

Now we return to our context, and let

$$\mathcal{U}=\mathcal{U}_{G/H}^{G/K}\colon (G/K,R)\text{-gr}\to(G/H,R)\text{-gr}$$

be the grade forgetting functor introduced in Section 2 and

$$\mathcal{F}=\mathcal{F}_{G/H}^{G/K}\colon (G/H,R)\to(G/K,R)\text{-gr}$$

its right adjoint.

**4.6. Proposition.** *Let $R$ be a $G$-graded ring and $K \leq H$ subgroups of $G$. If $M$ and $N$ are two objects of $(G/K,R)$-gr, then the following statements hold:*

*a) $M$ is small in $(G/H,R)$-gr if and only if $M$ is small in $(G/K,R)$-gr.*

*b) $M$ is $\mathcal{U}(N)$-small in $(G/H,R)$-gr if and only if $M$ is $\mathcal{F}(\mathcal{U}(N))$-small in $(G/K,R)$-gr.*

*Proof.* a) The necessity is obvious since we have that $\mathrm{Hom}_{(G/K,R)\text{-gr}}(M,N)$ is a subset of $\mathrm{Hom}_{(G/H,R)\text{-gr}}(M,N)$ for every $N \in (G/K,R)$-gr. The sufficiency follows by Theorem 4.4.b), applied to the functor $\mathcal{U}$ whose right adjoint $\mathcal{F}$ preserves direct sum.

b) By Theorem 4.2.a), we have that $M = \mathcal{U}(M)$ is $\mathcal{U}(N)$-small in $(G/H,R)$-gr if and only if $M$ is $\mathcal{F}(\mathcal{U}(N))$-small in $(G/K,R)$-gr.

*4.7. Remarks.* a) If $M$ is a $G$-graded $R$-module, then $M$ may be regarded as an object of $(G/K,R)$-gr via the functor

$$\mathcal{U}^{G/1}_{G/K}\colon R\text{-gr} \to (G/K,R)\text{-gr}.$$

Then statement a) of the above becomes:

*The object $M$ of $R$-gr is $\mathcal{U}^{G/K}_{G/H}(N)$-small in $(G/H,R)$-gr if and only if $M$ is $\mathcal{F}(\mathcal{U}(N))$-small in $(G/K,R)$-gr.*

? b) If $(\mathcal{U},\mathcal{F})$ is the pair of adjoint functors defined as above and if $N \in (G/K,R)$-gr then we have:

$$\mathcal{F}(\mathcal{U}(N)) = \bigoplus_{gK\in G/K} \Big( \bigoplus_{y \in {}^gH/{}^gK} N(g)_y \Big) = \bigoplus_{gK\in G/K} \Big( \bigoplus_{h^gK \in {}^gH/{}^gK} N_h gK \Big)$$

where $N(g)$ is the $g$-th suspension of $N$ and ${}^gK = gKg^{-1}$ [cf. M]

**4.8. Lemma.** *c. Let $M \in R$-gr and $N \in (G/H,R)$-gr. If $M$ is $N$-small in $R$-mod then*

$$\mathrm{HOM}_{G/H,R}(M,N) = \mathrm{Hom}_R(M,N)$$

*Proof..* As in the section 3 we associate to each $R$-morphism $f \in \mathrm{Hom}_R(M,N)$ and each $x = gH \in G/H$ a morphism $f_x \in \mathrm{HOM}_{G/H,R}(M,N)_x = \mathrm{Hom}_{(G/{}^gH,R)\text{-gr}}(M,N(g))$ defined by $f_x(m_h) = f_{gH}(M_h) = f(m_h)_{hgH}$ for every $h \in G, m_h \in M_h$. Conform th.3.??????. $\{f_x \mid x \in G/H\}$ is summable to $f$ in the finite topology i.e. $\sum_{x\in G/H} f_x = f$. Since every $f(m)$ has a unique (finite!) decomposition in homogeneous component, it follow that $supp\{f_x(m) \mid x \in G/H\}$ is finite. Thus we can define a mapping $g\colon M \to \bigoplus_{x\in G/H} N^x$, where $N^x \cong N$, by $g(m) = (f_x(m))_{x\in G/H}$ for every $m \in M$. Obviously, $g$ is an $R$-morphism and since $M$ is $N$-small in $R$-mod, there exists $x_1,\dots,x_n \in G/H$ such that $g(M) \subseteq \bigoplus_{i=1}^n N^{x_i}$. Therefore $f_x = 0$ for $x \in G/H\backslash\{x_1,\dots,x_n\}$ and $f = \sum_{i=1}^n f_{x_i}$. This show that $f \in \mathrm{Hom}_{G/H,R}(M,N)$.

**4.9. Corollary.** *Let $R$ be a $G$-graded ring and $K \leq H \leq G$. Let $M \in R$-gr and $N \in (G/K, R)$-gr. If $M$ is $N$-small in $R$-mod then*

$$\mathrm{HOM}_{G/H,R}(M, N) = \mathrm{HOM}_{G/K,R}(M, N)$$

The main result of this section is the following:

**4.10. Theorem.** *Let $R$ be a $G$-graded ring, $K \leq H \leq G$ two subgroups such that $H/K$ is infinite and $K$ is normal in $G$. Let $M \in R$-gr. Then the following statements are equivalent:*

*(i) $M$ is small in $(G/K, R)$-gr ($R$-mod, $R$-gr, $(G/H, R)$-gr)*

*(ii)$\mathrm{HOM}_{G/H,R}(M, N) = \mathrm{HOM}_{G/K,R}(M, N)$ for every $N \in (G/K, R)$-gr*

*Proof.* (i)$\Rightarrow$(ii) From prop.4.1. and corollary 4.7.

(ii)$\Rightarrow$(i) Let $M = \bigoplus_{g\in G} M_g \in R$-gr such that $\mathrm{HOM}_{G/H,R}(M, N) = \mathrm{Hom}_R(M, N)$ for every $N \in (G/H, R)$-gr. Since $K \trianglelefteq G$ we have for every $g \in G$ that ${}^gK = gKg^{-1} = K$ and the $g$-th suspension of $N$ noted $N(g)$ belongs to $(G/{}^gK, R)$-gr $= (G/K, R)$-gr. The $G/K$-grader of $N(h)$ is give as $N(h) = \bigoplus_{g\in[G/K]} N(h)_{gK}$ with $N(h)_{gK} = N_{ghK}$ and of $\bigoplus_{h\in[H/K]} N(h)$ as

$$\bigoplus_{h\in[H/K]} N(h) = \bigoplus_{g\in[G/K]} ( \bigoplus_{h\in[H/K]} N_{ghK})$$

Let $u \in \mathrm{Hom}_{(G/K,R)\text{-gr}}(M, \bigoplus_{h\in[H/K]} N(h)$ where $[H/K] \subseteq H$ is a transversely for $H/K$ (as we have seen $\bigoplus_{h\in[H/K]} N(h) \in (G/K, R)$-gr). Since every $n \in \bigoplus_{h\in[H/K]} N(h)$ has a unique finite decomposition $n = n(h_1)+\ldots+n(h_k)$ with $n(h_j) \in N(h_j), 1 \leq j \leq k$, we may define a $R$-morphism

$$t\colon \bigoplus_{h\in[H/K]} N(h) \to N \mathrm{by} t(n) = \sum_{i=1}^{k} n_(h_j) = \sum_{h\in[H/K} n(h).$$

In fact $t$ is the $R$-morphism $\bigoplus_{h\in[H/K]} N(h) \to N$ with all component $1_N$. Moreover

$$t( \bigoplus_{h\in[H/K]} N(h)_{gK}) = t( \bigoplus_{h\in[H/K]} N_{ghK}) \subseteq N_{gH}$$

, where $N_{gH}$ is the homogeneous component of $N$ see as $G/H$-graded. This show that $t \in \mathrm{HOM}_{G/H,R}(M, \bigoplus_{h\in[H/K]} N(h))$. We set $\bar{u} = t \circ u\colon M \to N$. Then

$$\bar{u} \in \mathrm{HOM}_{G/H,R}(M, N) = \mathrm{HOM}_{G/K,R}(M, N) = \bigoplus_{g\in[G/K]} \mathrm{HOM}_{G/K,R}(M, N)_{gK},$$

and so there exists $g_1K,\dots,g_sK \in G/K$ and $u_{g_i} \in \mathrm{HOM}_{G/K,R}(M,N)_{g_iK} = \mathrm{Hom}_{(G/K,R)\text{-gr}}(M,N(g_i)), 1 \le i \le s$ such that $\bar{u} = \sum_{i=1}^{s} u_{g_i}$. Thus $\bar{u}(M_g) \subseteq N(g_1)_{gK} + \dots N(g_s)_{gK} = N_{gg_1K} + \dots + N_{gg_sK}$ for each $g \in G$. It follows that if $m_g \in M_g$ for some $g \in G$ and $u(m_g) = (n_h)_{x\in[H/K]}$, with $n_h \in N(h)_{gK} = N_{ghK}$ then

$$\bar{u}(m_g) = \sum_{h\in[H/K]} n(h) \in N_{gg_1K} + \dots + N_{gg_sK}.$$

This shows that $n(h) = 0$ for each $h \in [H/K]\backslash\{gg_1,\dots,gg_s\}$. Consequently $\mathrm{Im}\, u \subseteq \bigoplus_{i=1}^{n} N(h_i)$ with $n \le s$ and this inclusion hold for an arbitrary morphism $u \in \mathrm{Hom}_{(G/K,R)\text{-gr}}(M,N)$.

Let now $f\colon M \to \bigoplus_{i\in\mathbb{N}} X_i$ be a morphism in $(G/K,R)$-gr. Let $A = \bigoplus i \in \mathbb{N}X_i$ and $N = \bigoplus_{g\in[G/K]} A(g) \in (G/K,R)$-gr. Then $N$ has the property that $N(g) \cong N$ in $(G/K,R)$-gr for each $g \in [G/K]$, in particular, for each $g = h \in [H/K]$. Since $H/K$ is infinite, we may assume that $\mathbb{N}$ is a subset of $H/K$ and we obtain a monomorphism in $(G/K,R)$-gr:

$$v\colon N^{(\mathbb{N})} \to \bigoplus_{h\in[H/K]} N(h).$$

We note by $\sigma\colon A \to N$ and $\rho_i\colon X_i \to A$ the canonical injections. Since $(G/K,R)$-gr is AB3, the morphism

$$w = \bigoplus_{i\in\mathbb{N}} (\sigma\circ\rho_i)\colon A \to N^{(\mathbb{N})}$$

is a monomorphism. We get a morphism in $(G/K,R)$-gr: $v\circ w\circ f\colon M \to \bigoplus_{h\in[H/K]} N(h)$. As we have seen above, $\mathrm{Im}(v\circ w\circ f) \subseteq \bigoplus_{i=1}^{n} N^(h_i)$ for some elements $h_1,\dots,h_n \in [H/K]$. Consequently $\mathrm{Im}\, f \subseteq \bigoplus i \in FX_i$, where $F = \mathbb{N}\cap\{h_1,\dots,h_n\}$ and hence we see that $M$ is small in $(G/K,R)$-gr.

The main result of this section is the following:

**4.10. Theorem.** *Let $R$ be a $G$-graded ring, $H \le G$ a subgroup such that $G/H$ is infinite and $M \in R$-gr. Then the following statements are equivalent:*

*(i) $M$ is small in $(G/H,R)$-gr ($R$-mod, $R$-gr)]*

*(ii) $\mathrm{HOM}_{G/H,R}(M,N) = \mathrm{Hom}_R(M,N)$ for every $N \in (G/H,R)$-gr.*

*Proof.* (i)$\Rightarrow$(ii) from prop.4.1. and lemma 4.5.

(ii)$\Rightarrow$(i) Let $M = \bigoplus_{g\in G} M_g \in R$-gr such that $\mathrm{HOM}_{G/H,R}(M,N) = \mathrm{Hom}_R(M,N)$ for every $N \in (G/H,R)$-gr. Let $u \in \mathrm{Hom}_{(G/H,R)\text{-gr}}(M, \bigoplus_{x\in G/H} N^x)$ where $N^x \cong N$, hence $u(M_g) \subseteq \bigoplus_{x\in G/H}(N^x)_g H$ for every $g \in G$. Since every $n \in \bigoplus_{x\in G/H} N^x$ has a unique finite decomposition $n = n_{x_1} + \dots + n_{x_k}$ with $n_{x_j} \in N^{x_j}, 1 \le j \le k$, we may define a mapping

$$t\colon \bigoplus_{x\in G/H} N^x \to N \text{by} t(n) = \sum_{i=1}^{k} n_{x_j} = \sum_{x\in G/H} n_x.$$

Obviously $t$ is a $R$-morphism. We set $\bar{u} = t\circ u\colon M \to N$. Then

$$\bar{u} \in \mathrm{Hom}_R(M,N) = \mathrm{HOM}_{G/H,R}(M,N) = \bigoplus_{y\in G/H} \mathrm{HOM}_{G/H,R}(M,N)_y,$$

and so there exists $y_1,\dots,y_s \in G/H$ and $u_{y_i} \in \mathrm{HOM}_{G/H,R}(M,N)_{y_i}, 1\le i\le s$ such that $\bar{u} = \sum_{i=1}^s u_{y_i}$. Thus $\bar{u}(M_g) \subseteq N_{gy_1}+\ldots+N_{gy_s}$ for each $g\in G$. It follows that if $m_h \in M_h$ for some $h\in G$ and $u(m_h) = (n_x)_{x\in G/H}$, with $n_x \in (N^x)_h H$ then

$$\bar{u}(m_h) = \sum_{x\in G/H} n_x \in N_{y_1}+\ldots+N_{y_s}$$

and $N_{y_i} = N_h^{x_i}H$, for some $x_i \in G/H, 1\le i\le s$. This show that $n_x = 0$ for each $x \in G/H\backslash\{x_1,\dots,x_s\}$. Consequently $\mathrm{Im}\, u \subseteq \bigoplus_{i=1}^s N^{x_i}$ and this inclusion holds for an arbitrary morphism $u\in \mathrm{Hom}_{(G/H,R)\text{-gr}}(M,N)$.

Now let $f\colon M \to \bigoplus i\in\mathbb{N}X_i$ be a morphism in $(G/H,R)$-gr. Let $A = \bigoplus i \in \mathbb{N}X_i$ and $N = \bigoplus_{x\in G/H} A^x \in (G/H,R)$-gr with $A^x \cong A$. Since $G/H$ is infinite, we may assume that $\mathbb{N}$ is a subset of $G/H$ and we obtain a monomorphism in $(G/H,R)$-gr:

$$v\colon N_{(\mathbb{N})} \to \bigoplus_{x\in G/H} N.$$

We note by $\sigma\colon A\to N$ and $\rho_i\colon X_i \to A$ the canonical injections. Since $(G/H,R)$-gr is AB3, the morphism

$$w = \bigoplus_{i\in\mathbb{N}} (\sigma\circ\rho_i)\colon A \to N^{\mathbb{N}}$$

is a monomorphism. We get a morphism in $(G/H,R)$-gr: $v\circ w\circ f\colon M\to \bigoplus_{x\in G/H} N^x$ with $N^x \cong N$. As we have seen above, $\mathrm{Im}(v\circ w\circ f) \subseteq \bigoplus_{i=1}^s N^{x_i}$ for some elements $x_1,\dots,x_s \in G/H$. Consequently $\mathrm{Im}\, f \subseteq \bigoplus i \in FX_i$, where $F = \mathbb{N}\cap\{x_1,\dots,x_s\}$ and hence we see that $M$ is small in $(G/H,R)$-gr.

**4.11. Corollary.** *Let $R$ be a $G$-graded ring and $K\le H\le G$ two subgroups such that $H/K$ is infinite. Then the following statements are equivalent:*

*(i) $M$ is small in $(G/H,R)$-gr ($R$-mod, $R$-gr).*

*(ii) $\mathrm{HOM}_{G/H,R}(M,N) = \mathrm{HOM}_{G/K,R}(M,N)$ for every $N \in (G/K,R)$-gr.*

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**2.5.** An another functor which we will use is the "suspension" functor:

$$\mathcal{S}_G/H^g\colon (G/H,R)\text{-gr} \to (G/{}^gH)\text{-grdefined by } \mathcal{S}_G/H^g(N) = N(g), g\in G,$$

where ${}^gH = gHg^{-1}$ and $N(g) = \bigoplus_{x\in G/{}^gH} N(G)_x$ with $N(g)_x = N_{xg}$.

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**3.6. Theorem.** *Let $R$ be a $G$-graded ring and $H \leq G$. Let $M \in R$-gr and $N \in (G/H, R)$-gr. With the above notations we have:*

*(i) The family $\{f_x \mid x \in G/H\}$ is summable to $f$ in the finite topology with $f_x$ uniquely determined with the properties $f_x \in \mathrm{HOM}_{G/H,R}(M,N)_x$ and $\sum_{x\in G/H} f_x = f$.*

*(ii) $\mathrm{Hom}_R(M,N)$ is the completion of $\mathrm{HOM}_{G/H,R}(M,N)$ in the finite topology.*

*Proof.* (i) We restrict at neighborhoods of the form $V(f, m_g)$ with $m_g \in M_g$ for some $g \in G$, a homogeneous element. Lte $f(m_g) = n_{x_1} + \ldots + n_{x_k}$ with $n_{x_j} \in N_{x_j}$ the decomposition of $f(m_g)$ in the homogeneous components. We set $J_0 = \{x_1, \ldots x_k\}$. Then $J_0$ is a finite subset of $G/H$ and $f(m_g) = (\sum_{x\in J_0} f_x(m_g)$ hence $\sum_{x\in J_0} f_x \in V(f, m_g)$. Since for each $x \in G/K\backslash J_0$ we have $f(m_g) = 0$, it follows that, for every finite subset $J$ of $I$, containing $J_0$, $f(m_g) = (\sum_{x\in J} f_x(m_g)$, therefore $\sum_{x\in J} f_x \in V(f, m_g)$. Consequently, $\sum_{x\in G/H} f_x = f$ in the finite topology.

For the uniqueness we assume that $g_x \mid x \in G/H$ is another family of morphisms such that $g_x \in \mathrm{HOM}_{G/H,R}(M,N)_x$ and $\sum_{x\in G/H} g_x = f$. If $f_{x_0} \neq g_{x_0}$ for some $x_0 \in G/H$ then there exit $m_g \in M_g$ such that $f_{x_0} \neq g_{x_0}$. We consider the neighborhood $V(f, m_g)$ of $f$ in the finite topology. Then we may find $J_0 \in \mathcal{P}_f(G/H)$ where $\mathcal{P}_f(G/H)$ is the set of all finite subsets of $G/H$, whiththe property that for every $J \in \mathcal{P}_f(G/H)$ for which $J_0 \in J$ we have $\sum_{x\in G/H} f_x \in V(f, m_g)$ and $\sum_{x\in G/H} g_x \in V(f, m_g)$. We set $J = J_0 \cup \{x_0\}$. Then $J \in \mathcal{P}_f(G/H)$ and $(\sum_{x\in J} f_x(m_g) = f(m_g) = (\sum_{x\in J} g_x(m_g)$ and from the uniqueness of the decomposition in homogeneous elements it follows that $f_{x_0}(m_g) = g_{x_0}(m_g)$, which is a contradiction.

(ii) Given $f$ belonging to $\mathrm{Hom}_R(M,N)$ Whit the above notations, for some $J \in \mathcal{P}_f(G/H)$ we have

$$\sum_{x\in J} f_x \in \bigoplus_{x\in G/H} \mathrm{HOM}_{G/H,R}(M,N)_x = \mathrm{HOM}_{G/H,R}(M,N)$$

and the result (i) implies $\mathrm{HOM}_{G/H,R}(M,N) \cap V(f,m) \neq 0$. Hence $\mathrm{HOM}_{G/H,R}(M < N)$ is dense in $\mathrm{Hom}_R(M,N)$ on the finite topology. But $\mathrm{Hom}_R(M,N)$ is a complete Hausdorff topological space in the finite topology, consequently, it is the completion of $|HOM_{G/H,R}(M,N)$.

*3.7. Remark.* If $1 = H \leq G$ then $(G/1, R)$-gr $= R$-gr and and the above theorem is just the theorem 1.2. of [GN].

**3.8. Corollary.** *If $R$ is $G$-graded ring and $K \leq H \leq G$ are two subgroups then $\mathrm{HOM}_{G/K,R}(M,N)$ is dense in $\mathrm{HOM}_{G/H,R}(M,N)$ on the finite topology.*

**3.9. Proposition.** *Let $R$ be a $G$-graded ring, $M \in R$-gr, and $N \in (G/K, R)$-gr where $K \leq H \leq G$ are two subgroups. Thus each $f \in \mathrm{HOM}_{G/H,R}(M,N)_{gH}, (g \in G)$ may be write as $f = \sum_{z \in {}^gH/{}^gK} f_z$ where $f_z \in \mathrm{HOM}_{{}^gH/{}^gK,R}(M,N)_z$*

*Proof.* Let

$$f \in \mathrm{HOM}_{G/H,R}(M,N)_{gH} = \mathrm{Hom}_{(G/{}^gH,R)\text{-gr}}(M, N(g)) = \mathrm{HOM}_{G/{}^gH,R}(M,N)_{{}^gH}$$

Since $f \in \mathrm{Hom}_R(M,N)$ th. 3.6. give the relation $f = \sum_{h \in [G/{}^gK]} f_{h^gK}$ where $f_{h^gK} \in \mathrm{HOM}_{G/{}^gK,R}(M,N)$ and $[G/{}^gK] \subseteq G$ is a transversely for $G/{}^gK$. Moreover, we have for each $k \in G$: $f(M_k) \subseteq \mathcal{U}(N(g))_{{}^kH} = \bigoplus_{x \in G/{}^gK} N(g)_x$ and $f_{h^gK} \subseteq N_{kh^gK}$. Since $\mathrm{HOM}_{G/{}^gH,R}(M,N)$ is a Hausdorff topological space and $f = \sum_{h \in [G/{}^gK]} f_{h^gK}$ it follows that $N_{kh^gK} \subseteq \bigoplus_{x \in G/{}^gK} N(g)_x$. Thus $f_{h^gK}(M_k) \neq)$ implies $kh^gK \subseteq k^gH$

**3.10. Corollary.** *With the notations from the prop. 3.8., if H/K is a finite set then*

$$\mathrm{HOM}_{G/H,R}(M,N) = \mathrm{HOM}_{G/K,R}(M,N)$$

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The main result of this section is the following:

**4.8. Theorem.** *Let $R$ be a $G$-graded ring, $K \leq H \leq G$ two subgroups such that $H/K$ is infinite and $K$ is normal in $G$. Let $M \in R$-gr. Then the following statements are equivalent:*

*(i) $M$ is small in $(G/K,R)$-gr ($R$-mod, $R$-gr, $(G/H,R)$-gr)*

*(ii)*$\mathrm{HOM}_{G/H,R}(M,N) = \mathrm{HOM}_{G/K,R}(M,N)$ *for every* $N \in (G/K,R)$-gr

*Proof.* (i)$\Rightarrow$(ii) From prop.4.1. and corollary 4.7.

(ii)$\Rightarrow$(i) Let $M = \bigoplus_{g \in G} M_g \in R$-gr such that $\mathrm{HOM}_{G/H,R}(M,N) = \mathrm{Hom}_R(M,N)$ for every $N \in (G/H,R)$-gr. Since $K \trianglelefteq G$ we have for every $g \in G$ that ${}^gK = gKg^{-1} = K$ and the $g$-th suspension of $N$ noted $N(g)$ belongs to $(G/{}^gK,R)$-gr $= (G/K,R)$-gr. The $G/K$-grader of $N(h)$ is give as $N(h) = \bigoplus_{g \in [G/K]} N(h)_{gK}$ with $N(h)_{gK} = N_{ghK}$ and of $\bigoplus_{h \in [H/K]} N(h)$ as

$$\bigoplus_{h \in [H/K]} N(h) = \bigoplus_{g \in [G/K]} \Big( \bigoplus_{h \in [H/K]} N_{ghK} \Big)$$

Let $u \in \mathrm{Hom}_{(G/K,R)\text{-gr}}(M, \bigoplus_{h \in [H/K]} N(h)$ where $[H/K] \subseteq H$ is a transversely for $H/K$ (as we have seen $\bigoplus_{h \in [H/K]} N(h) \in (G/K,R)$-gr). Since every $n \in \bigoplus_{h \in [H/K]} N(h)$

has a unique finite decomposition $n = n(h_1)+\ldots+n(h_k)$ with $n(h_j) \in N(h_j), 1 \le j \le k$, we may define a $R$-morphism

$$t\colon \bigoplus_{h\in[H/K]} N(h) \to N \text{by} t(n) = \sum_{i=1}^{k} n_(h_j) = \sum_{h\in[H/K} n(h).$$

In fact $t$ is the $R$-morphism $\bigoplus_{h\in[H/K]} N(h) \to N$ with all component $1_N$. Moreover

$$t(\bigoplus_{h\in[H/K]} N(h)_{gK}) = t(\bigoplus_{h\in[H/K]} N_{ghK}) \subseteq N_{gH}$$

, where $N_{gH}$ is the homogeneous component of $N$ see as $G/H$-graded. This show that $t \in \mathrm{HOM}_{G/H,R}(M, \bigoplus_{h\in[H/K]} N(h))$. We set $\bar{u} = t \circ u\colon M \to N$. Then

$$\bar{u} \in \mathrm{HOM}_{G/H,R}(M,N) = \mathrm{HOM}_{G/K,R}(M,N) = \bigoplus_{g\in[G/K]} \mathrm{HOM}_{G/K,R}(M,N)_{gK},$$

and so there exists $g_1K, \ldots, g_sK \in G/K$ and $u_{g_i} \in \mathrm{HOM}_{G/K,R}(M,N)_{g_iK} = \mathrm{Hom}_{(G/K,R)\text{-gr}}(M, N(g_i)), 1 \le i \le s$ such that $\bar{u} = \sum_{i=1}^{s} u_{g_i}$. Thus $\bar{u}(M_g) \subseteq N(g_1)_{gK} + \ldots N(g_s)_{gK} = N_{gg_1K} + \ldots + N_{gg_sK}$ for each $g \in G$. It follows that if $m_g \in M_g$ for some $g \in G$ and $u(m_g) = (n_h)_{x\in[H/K]}$, with $n_h \in N(h)_{gK} = N_{ghK}$ then

$$\bar{u}(m_g) = \sum_{h\in[H/K]} n(h) \in N_{gg_1K} + \ldots + N_{gg_sK}.$$

This shows that $n(h) = 0$ for each $h \in [H/K]\backslash\{gg_1, \ldots, gg_s\}$. Consequently $\mathrm{Im}\, u \subseteq \bigoplus_{i=1}^{n} N(h_i)$ with $n \le s$ and this inclusion hold for an arbitrary morphism $u \in \mathrm{Hom}_{(G/K,R)\text{-gr}}(M,N)$.

Let now $f\colon M \to \bigoplus_{i\in\mathbb{N}} X_i$ be a morphism in $(G/K,R)$-gr. Let $A = \bigoplus i \in \mathbb{N} X_i$ and $N = \bigoplus_{g\in[G/K]} A(g) \in (G/K,R)\text{-gr}$. Then $N$ has the property that $N(g) \cong N$ in $(G/K,R)$-gr for each $g \in [G/K]$, in particular, for each $g = h \in [H/K]$. Since $H/K$ is infinite, we may assume that $\mathbb{N}$ is a subset of $H/K$ and we obtain a monomorphism in $(G/K,R)$-gr:

$$v\colon N^{(\mathbb{N})} \to \bigoplus_{h\in[H/K]} N(h).$$

We note by $\sigma\colon A \to N$ and $\rho_i\colon X_i \to A$ the canonical injections. Since $(G/K,R)$-gr is AB3, the morphism

$$w = \bigoplus_{i\in\mathbb{N}} (\sigma \circ \rho_i)\colon A \to N^{(\mathbb{N})}$$

is a monomorphism. We get a morphism in $(G/K,R)$-gr: $v \circ w \circ f\colon M \to \bigoplus_{h\in[H/K]} N(h)$. As we have seen above, $\mathrm{Im}(v \circ w \circ f) \subseteq \bigoplus_{i=1}^{n} N^(h_i)$ for some elements $h_1, \ldots, h_n \in [H/K]$. Consequently $\mathrm{Im}\, f \subseteq \bigoplus i \in F X_i$, where $F = \mathbb{N} \cap \{h_1, \ldots, h_n\}$ and hence we see that $M$ is small in $(G/K,R)$-gr.

**3.9. Proposition.** *With the notations from the prop. 3.8., if $H/K$ is a finite set then*

$$\mathrm{HOM}_{G/H,R}(M,N)=\mathrm{HOM}_{G/K,R}(M,N)$$

.

*Proof.* For begin let $f\in\mathrm{HOM}_{G/H,R}(M,N)_{gH}$ and $\{h_1,\dots,h_n\}=[H/K]$ be a transversely for $H/K$. Since $H=\bigcup_{h\in[H/K]}hK$ we have, for a fixed $gH\in G/H$, $gH=\bigcup_{h\in[H/K]}ghK=gh_1K\cup\ldots\cup gh_nK$, and $G/K=\{ghK\mid g\in[G/H],h\in[H/K]\}$. Thus

$$N=\bigoplus_{x\in G/K}N_x=\bigoplus_{g\in[G/K]}(\bigoplus_{h\in[H/K]}N_{ghK})=\bigoplus_{g\in[G/H]}(\bigoplus_{i=1}^{n}N_{gh_iK})=\bigoplus_{i=1}^{n}(\bigoplus_{g\in[G/H]}N_{gh_iK}).$$

We consider the canonical projections $p_n\colon N\to\bigoplus_{g\in[G/H]}N_{gh_iK}$ and the composition $p_n\circ f\colon M\to\bigoplus_{g\in[G/H]}N_{gh_iK}$. Since $f(M_k)\subseteq N_{kgH}$ and for each $kH\in G/H$ there exist a unique $i\in\{1,\dots,n\}$ such that $gh_iK\subseteq kgH$. Thus $f$ may be write as $\sum_{i=1}^{n}(f\circ p_n)$ with $f\circ p_n\in\mathrm{HOM}_{G/K,R}(M,N)$. Now the general case when $f\in\mathrm{HOM}_{G/H,R}(M,N)$ follow from the fact that $f$ may be write as a finite sum of morfisms belonging to $\mathrm{HOM}_{G/H,R}(M,N)_{gH}$ with $gH$ ranges over $G/H$.

## References


3. E.C. Dade, *Group-Graded Rings and Modules*, Math. Z. **174** (1980), 241-262.
5. E.C. Dade, *Clifford theory for group graded rings*, J. Reine Angew. Math. **369** (1986), 40-86.
14. C. Năstăsescu, Ş. Raianu and F. van Oystaeyen, *Graded modules over $G$-sets*, Math. Z. **203** (1990), 605-627.
15. C. Năstăsescu, L. Shaoxue and F. van Oystaeyen, *Graded modules over $G$-sets II*, Math. Z. **207** (1991), 341-358.
17. C. Năstăsescu and F. van Oystaeyen, *Graded Ring Theory*, North-Holland, 1982.
22. Bo Stenstrom, *Rings of Quotients*, Springer-Verlag, 1975.